Continuous probability distribution

A probability density function (p.d.f) is continuous if it takes on values between an interval.

Properties of a continuous probability density functions

(i) $\int f(x)dx = 1$
(ii) f (x) ≥ 0

Example 1

A random variable X of continuous p.d.f is given by $f(x) = \begin{cases} kx & 0 \leq x \leq 5 \\ 0 & elsewhere \end{cases}$

Find the value of k

Solution

$\int_0^5 kxdx = 1$

$k\left[\frac{x^2}{2}\right]_0^5 = 1$

$k\left(\frac{5^2}{2} - \frac{0^2}{2}\right) = 1$

$k\frac{25}{2} = 1$

$k = \frac{2}{25}$

Example 2

A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx & 0 \leq x \leq 2 \\ 2k(x-1), & 2 \leq x \leq 4 \\ 0, & elsewhere \end{cases}$

Solution

$\int_0^2 kxdx + \int_2^4 2k(x-1)dx = 1$

$k\left[\frac{x^2}{2}\right]_0^2 + 2k\left[\frac{x^2}{2} - x\right]_2^4 = 1$

$\left(\frac{2^2}{2} - \frac{0^2}{2}\right) + 2k\left\{\left(\frac{4^2}{2} - 4\right) - \left(\frac{2^2}{2} - 2\right)\right\} = 1$

2k + 8k = 1; k= $\frac{1}{10}$

Sketching f(x)

- find the initial and final points of f(x)
- join the initial and final points of f(x) using a line or curve.

Note

- A line is in the form of y = mx + c
- A curve has a power of x being 2 and above or fractional power e.g. $y = x^2$.
- A curve has a positive coefficient of x^2 has a minimum turning point while a curve with a negative coefficient has a maximum turning point

Example 3

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} kx & 0 \le x \le 3 \\ 0, & elsewhere \end{cases}$

Find the value of the constant k and sketch f(x)

Solution

$\int_0^3 kxdx = 1$

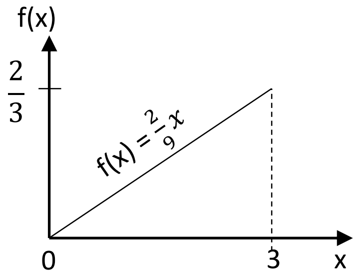
$k\left[\frac{x^2}{2}\right]_0^3 = 1$

$k\left(\frac{3^2}{2} - \frac{0^2}{2}\right) = 1$

$k = \frac{2}{9}$

When x = 0, $f(x) = \frac{2}{9}\ x\ 0 = 0$

When x = 3, $f(x) = \frac{2}{9}\ x\ 3 = \frac{2}{3}$



Example 4

A random variable X of continuous p.d.f is given by $f(x) = \begin{cases} kx, & 0 \le x \le 3 \\ k(6-x), & 3 \le x \le 6 \\ 0, & elsewhere \end{cases}$

Find the value of the constant k and sketch x

Solution

$\int_0^3 kxdx + \int_3^6 k(6-x)dx = 1$

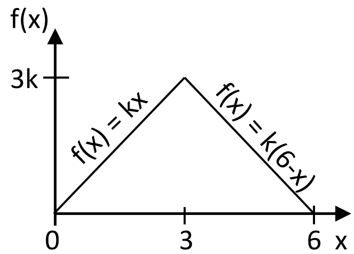
$k\left[\frac{x^2}{2}\right]_0^3 + k\left[6x - \frac{x^2}{2}\right]_3^6 = 1$

$k = \frac{1}{9}$

When x = 0, f(x) = k(0) = 0

When x = 3, f(x) = k(3) = 3k

When x = 6, f(x) = k(6-6) = 0



Example 5

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} k(x+2), & -2 \le x \le 0 \\ k(2-x), & 0 \le x \le 2 \\ 0, & elsewhere \end{cases}$

Find the value of k and sketch f(x)

$\int_{-2}^0 k(x+2)dx + \int_0^2 k(2-x)dx = 1$

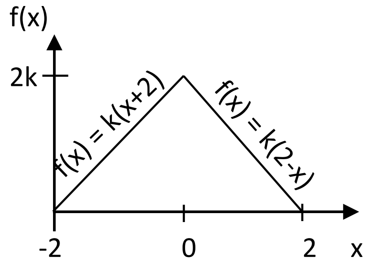
$k\left[\frac{x^2}{2} + 2x\right]_{-2}^0 + k\left[2x - \frac{x^2}{2}\right]_0^2 = 1$

$k = \frac{1}{4}$

When x = -2, f(x) = k(-2+2) = 0

When x = 0, f(x) = k(0 +2) = 2k

When x = 2, f(x) = k(2-2) = 0



Finding Probabilities

Example 6

A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx & 0 \le x \le 6 \\ 0, & elsewhere \end{cases}$

Find

(i) the value of k (ii) P(X > 4) (iii) P(X< 3) (iv) P(1< x < 3) (v) $P\left(X > 2 / X \le 4\right)$

Solution

(i) $\int_0^6 kxdx = 1$

$k\left[\frac{x^2}{2}\right]_0^6 = k\left[\frac{6^2}{2} - \frac{0^2}{2}\right] = 1$

k = $\frac{1}{18}$

(ii) P(X > 4) = $\frac{1}{18}\int_4^6 xdx = 1$

$= \frac{1}{18}\left[\frac{x^2}{2}\right]_4^6 = \frac{1}{18}\left[\frac{6^2}{2} - \frac{4^2}{2}\right] = \frac{5}{9} = 0.5556$

(iii) P(X < 3) = $\frac{1}{18}\int_0^3 xdx = 1$

$= \frac{1}{18}\left[\frac{x^2}{2}\right]_0^3 = \frac{1}{18}\left[\frac{3^2}{2} - \frac{0^2}{2}\right] = \frac{1}{4} = 0.25$

(iii) 1 < x < 3) = $\frac{1}{18}\int_1^3 xdx = 1$

$= \frac{1}{18}\left[\frac{x^2}{2}\right]_1^3 = \frac{1}{18}\left[\frac{3^2}{2} - \frac{1^2}{2}\right]$

$= \frac{2}{9} = 0.2222$

(iv) $P\left(X > 2 / X \le 4\right) = \frac{P(X>2 \cap X\le 4}{P(X\le 4)} = \frac{P(2<X<4)}{P(X\le 4)} = \frac{\frac{1}{18}\int_2^4 xdx=1}{\frac{1}{18}\int_0^4 xdx=1} = \frac{3}{4}$

Example 7

A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx(6-x) & 0 \le x \le 6 \\ 0, & elsewhere \end{cases}$

Find the (i) value of k and sketch f(x) (ii) P(X ≥ 5)

(i) $\int_0^6 kx(6-x)dx = 1$

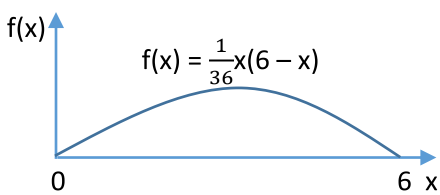
$k\left[3x^2 - \frac{x^3}{3}\right]_0^6 = k\left[\left(3x\ 6^2 - \frac{6^3}{3}\right) - \left(3\ x\ 0^2 - \frac{0^3}{3}\right)\right] = 1$

k = $\frac{1}{36}$

When x = 0, f(x) = $\frac{1}{36}(0)(6-0) = 0$

When x = 6, f(x) = $\frac{1}{36}(6)(6-6) = 0$

Sketch



(ii) P(X ≥ 5) = $\frac{1}{36}\int_5^6 x(6-x)dx$

$= \frac{1}{36}\left[3x^2 - \frac{x^3}{3}\right]_5^6 = \frac{1}{36}\left[\left(3x\ 6^2 - \frac{6^3}{3}\right) - \left(3\ x\ 5^2 - \frac{5^3}{3}\right)\right]$ = 0.074

Example 8

A random variable of continuous p.d.f is given by $f(x) = \begin{cases} kx^2 & 0 \le x \le 4 \\ 0, & elsewhere \end{cases}$

Find (i) value of k and sketch f(x)   (ii) P(1 ≤ x ≤ 3)

Solution

(i) $\int_0^4 kx^2 dx = 1$

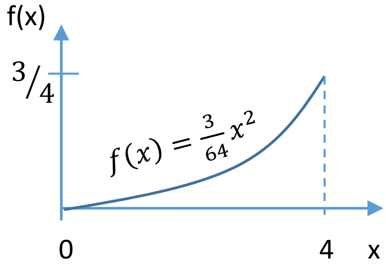
$k\left[\frac{x^3}{3}\right]_0^4 = k\left[\frac{4^3}{3} - \frac{0^3}{3}\right] = 1$

k = $\frac{3}{64}$

When x = 0, f(x) = $\frac{3}{64}0^2 = 0$

When x = 4, f(x) = $\frac{3}{64}4^2 = \frac{3}{4}$

Sketch



(ii) P(1 ≤ x ≤ 3)= $\frac{3}{64}\int_1^3 kx^2 dx = 1$

$= \frac{3}{64}\left[\frac{x^3}{3}\right]_1^3 = \frac{3}{64}\left[\frac{3^3}{3} - \frac{1^3}{3}\right] = 0.4063$

Example 9

A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(x^2+1) & 0 \le x \le 3 \\ 0, & elsewhere \end{cases}$

Find (i) value of k and sketch f(x)   (ii) P(1 ≤ x ≤ 3)

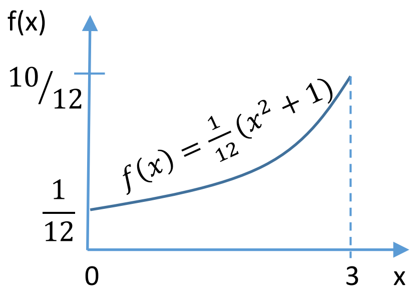
(i) $\int_0^3 k(x^2+1)dx = 1$

$k\left[\frac{x^3}{3} + x\right]_0^3 = k\left[\left(\frac{3^3}{3} + 3\right) - \left(\frac{0^3}{3} + 0\right)\right] = 1$

k = $\frac{1}{12}$

When x = 0, f(x) = $\frac{1}{12}(0^2+1) = \frac{1}{12}$

When x = 3, f(x) = $\frac{1}{12}[3^2+1] = \frac{10}{12}$



(ii) P(1 ≤ x ≤ 3)

$\frac{1}{12}\int_1^3 (x^2+1)dx = \frac{1}{12}\left[\frac{x^3}{3} + x\right]_1^3 = \frac{1}{12}\left[\left(\frac{3^3}{3} + 3\right) - \left(\frac{1^3}{3} + 1\right)\right] = 0.8889$

Example 10

A random variable X of continuous p.d.f is given by $f(x) = \begin{cases} k, & 0 \le x \le 2 \\ k(2x-3), & 2 \le x \le 3 \\ 0, & elsewhere \end{cases}$

Find (i) value of k and sketch f(x) (ii) P(X<1) (ii) P(X > 2.5) (iv) $\left(0 \le X \le 2 / X \ge 1\right)$

Solution

$\int_0^2 kdx + \int_2^3 k(2x-3)dx = 1$

$k[x]_0^2 + k[x^2 - 3x]_2^3 = 1$

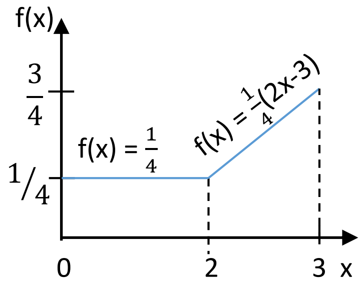
k = $\frac{1}{4}$

When x = 0, f(x) = k = $\frac{1}{4}$

When x = 2, f(x) = k = $\frac{1}{4}$

When x = 3, f(x) = $\frac{1}{4}$ (2 x3 - 3)

= $\frac{3}{4}$



(ii) P(X<1) $=\frac{1}{4}\int_0^1 dx = \frac{1}{4}[x]_0^1 = \frac{1}{4}$

(iii) P(X > 2.5) $= \frac{1}{4}\int_{2.5}^3 (2x-3)dx = \frac{1}{4}[x^2 - 3x]_{2.5}^3 = 0.3125$

(iv) $P\left(0 \le X \le 2 / X \ge 1\right) = \frac{P(0 \le X \le 2)}{P(X \ge 1)} = \frac{P((0 \le X \le 2) \cap (X \ge 1))}{P(X \ge 1)} = \frac{\frac{1}{4}\int_1^2 dx}{\frac{1}{4}\int_1^2 dx + \frac{1}{4}\int_2^3 (2x-3)dx} = \frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{2}} = \frac{1}{3}$

Example 11

A random variable X of continuous p.d.f is given by $f(x) = \begin{cases} k(x+2)^2, & -2 \le x \le 0 \\ 4k, & 0 \le x \le \frac{4}{3} \\ 0, & elsewhere \end{cases}$

Find

(i) the value of the constant k and sketch f(x)

(ii) P(-1 < x< 1) (iii) P(X > 1)

Solution

$\int_{-2}^0 k(x+2)^2 dx + \int_0^2 4kdx = 1$

$k\left[\frac{(x+2)^3}{3}\right]_{-2}^0 + 4k[x]_0^2 = 1$

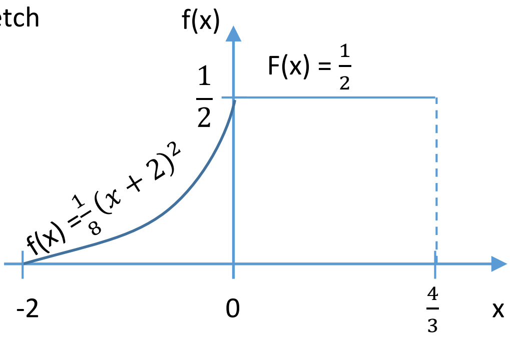
k = $\frac{1}{8}$

When x = -2, f(x) = $\frac{1}{8}$ (-2 + 2)^2 = 0

When x = 0, f(x) = $\frac{1}{8}$ (0+2)^2 = $\frac{1}{2}$

When x = $\frac{4}{3}$, f(x) =4 x $\frac{1}{8}$ = $\frac{1}{2}$

Sketch



(ii) P(-1 < x< 1) $= \int_{-1}^0 k(x+2)^2 dx + \int_0^1 4kdx$

$= \frac{1}{8}\left[\frac{(x+2)^3}{3}\right]_{-12}^0 + 4x\frac{1}{8}[x]_0^1 = \frac{7}{24} + \frac{1}{2} = \frac{19}{24}$

(iii) P(X > 1) $= \int_0^{\frac{4}{3}} 4kdx = 4x\frac{1}{8}[x]_1^{4/3} = \frac{1}{6}$

Finding the constant k from a sketch graph

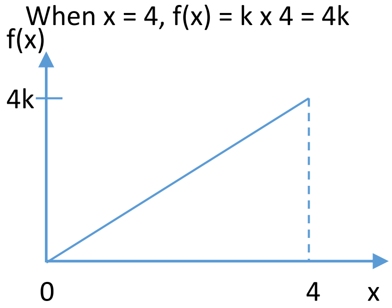
Example 12

A random variable X of continuous p.d.f is given by $f(x) = \begin{cases} kx & 0 \le x \le 4 \\ 0, & elsewhere \end{cases}$

(a) Sketch and find the value of constant k
(b) Find (i) P(X ≤ 1)   (ii) P(1 < x < 2)

Solution

(a) When x = 0, f(x) = k x 0 = 0
When x = 4, f(x) = k x 4 = 4k



Area under the curve = $\frac{1}{2}$ $x\ 4\ x\ 4k = 1$

k = $\frac{1}{8}$

(b)(i) P(X ≤ 1) = $\frac{1}{8}\int_0^1 xdx$ =$\frac{1}{8}\left[\frac{x^2}{2}\right]_0^1$

=$\frac{1}{8}\left(\frac{1^2}{2} - \frac{0^2}{2}\right)$ = $\frac{1}{16}$

(ii) P(1 < x < 2) = $\frac{1}{8}\int_1^2 xdx$ =$\frac{1}{8}\left[\frac{x^2}{2}\right]_1^2$

=$\frac{1}{8}\left(\frac{2^2}{2} - \frac{1^2}{2}\right)$ = $\frac{3}{16}$

Example 13

A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 0 \le x \le 2 \\ k(4-x), & 2 \le x \le 4 \\ 0, & elsewhere \end{cases}$

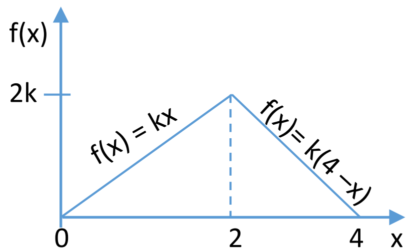
(a) Sketch f(x) and find the value of k
(b) Find (i) P(X< 1)   (ii) P(X >3)   (iii) P(1 ≤ x≤ 3)   (iv) $P\left(X \ge 1 / X \le 3\right)$

Solution

When x = 0, f(x) = $k(0)$ = 0

When x = 2, f(x) = $k\ x\ 2 = 2k$

When x = 4, f(x) =k (4 – 4) = 0



Area under the curve = $\frac{1}{2}$ $x\ 4\ x\ 2k = 1$

k = $\frac{1}{4}$

(b)(i) P(X< 1) = $\frac{1}{4}\int_0^1 xdx$ =$\frac{1}{4}\left[\frac{x^2}{2}\right]_0^1$

=$\frac{1}{4}\left(\frac{1^2}{2} - \frac{0^2}{2}\right)$ = $\frac{1}{8}$

(ii) P(X >3) = $\frac{1}{4}\int_3^4 (4-x)dx$

=$\frac{1}{4}\left[4x - \frac{x^2}{2}\right]_3^4$ = 0.125

(iii) P(1 ≤ x≤ 3)=$\frac{1}{4}\int_1^2 xdx + \frac{1}{4}\int_2^3 4 - xkdx$

= $\frac{1}{4}\left[\frac{x^2}{2}\right]_1^2$ + $\frac{1}{4}\left[4x - \frac{x^2}{2}\right]_2^3$ = $\frac{3}{4}$

(iv) $P\left(X \geq 1 / X \leq 3\right) = \frac{X\geq 1 \cap X\leq 3}{X\leq 3} = \frac{P(1\leq x \leq 3}{X\leq 3} = \frac{3/4}{\frac{1}{4}\int_0^2 xdx+\frac{1}{4}\int_2^3 4-xkdx} = \frac{3}{4}\Big/\frac{7}{8} = \frac{6}{7}$

Example 14

A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k, & 2 \leq x \leq 3 \\ k(x-2), & 3 \leq x \leq 4 \\ 0, & elsewhere \end{cases}$

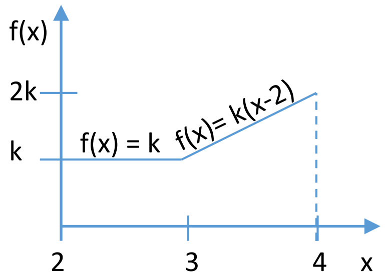
Find (i) the value of k and sketch the graph (ii) P($|X - 2.5| > 0.5$) (iii) P($|X - 2.5| < 0.5$)

Solution

(i) When x = 2, f(x) = k

When x = 3, f(x) = k

When x = 4, f(x) =k (4 – 2) = 2k



Area under the curve = 1 x k + $\frac{1}{2}(k + 2k)x\ 1 = 1$

k = $\frac{2}{5}$

(ii) P($|X - 2.5| > 0.5$) = P(-0.5 < X-2.5<0.5)

= P(2 < X <3)

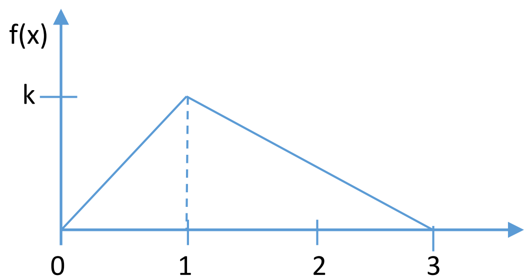
=$\frac{2}{5}\int_2^3 dx = [x]_2^3$

= $\frac{2}{5}$

Finding p.d.f from a sketch graph

Example 15

A random variable X of a continuous p.d.f is given by



(a) Area = 1 = $\frac{1}{2}\ x\ 3\ x\ k$

K = $\frac{2}{3}$

(b) Find f(x)

Let f(x) = y

For interval: $0 \leq x\ \leq 1$ coordinates are (0, 0) and (1, k)

grad = $\frac{y-0}{x-0}$ = $\frac{\frac{2}{3}-0}{1-0}$

y = $\frac{2}{3}x$

For interval 1≤ x≤ 3

Coordinates are (3,0) and (1, k)

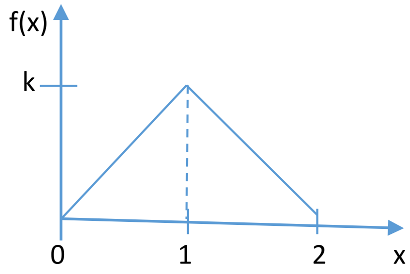
grad = $\frac{y-0}{x-3}$ = $\frac{\frac{2}{3}-0}{1-3}$

y = $-\frac{1}{3}(x-3)$

$$f(x) = \begin{cases} \frac{2}{3}x, & 0 \leq x \leq 1 \\ \frac{1}{3}(x-3), & 1 \leq x \leq 3 \\ 0, & elsewhere \end{cases}$$

Example 16

A continuous random variable X has a probability density function (p.d.f) f(x) as shown in the graph below



(a) Find the
   (i) value of k
   (ii) expression for the probability density function
(b) Calculate the
   (i) The mean
   (ii) P(X<1.5/X> 0.5)

Solution

(i) Area under the graph = 1

$\frac{1}{2}\ x\ 2\ x\ k = 1;$ k = 1

(ii) Let f(x) = y

For interval: $0 \leq x \ \leq 1$ coordinates are (0, 0) and (1, k)

grad = $\frac{y-0}{x-0}$ = $\frac{1-0}{1-0}$

y = x

For interval: $1 \leq x \ \leq 2$ coordinates are

(1, k) and (2, 0)

grad = $\frac{y-1}{x-1}$ = $\frac{0-1}{2-1}$

y = $2 - x$

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ (2-x), & 1 \leq x \leq 2 \\ 0, & elsewhere \end{cases}$$

(b)(i) E(X) =$\sum xf(x)$

$= \int_0^1 x.xdx + \int_1^2 x(2-x)dx$

$=\left[\frac{x^3}{3}\right]_0^1 + \left[x^2 - \frac{x^3}{3}\right]_1^2$

$=\left(\frac{1}{3} - 0\right) + \left[\left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)\right]$

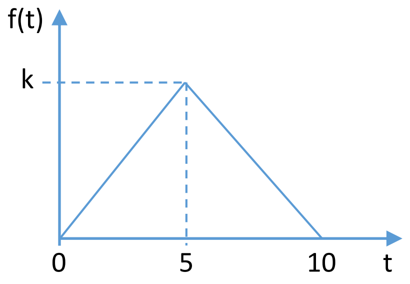
$=\frac{1}{3} + \frac{4}{3} - \frac{2}{3} = 1$

(b)(ii) P(X<1.5/X> 0.5) = $\frac{P(x<1.5\ \cap x>0.5)}{P(X>0.5)} = \frac{P(0.5<x\ 1.5)}{P(X>0.5)} = \frac{\int_{0.5}^{1} xdx+\int_{1}^{1.5}(2-x)dx}{1-\int_{0}^{0.5} xdx}$

$$= \frac{\left[\frac{x^2}{2}\right]_{0.5}^{1} + \left[2x - \frac{x^2}{2}\right]_{1}^{1.5}}{1-\left[\frac{x^2}{2}\right]_{0}^{0.5}} = 0.8751$$

Example 17

The departure time T of pupils from a certain day primary school can be modelled as in the diagram below, where t is the time in minutes after the final bell at 5.00pm



Determine the

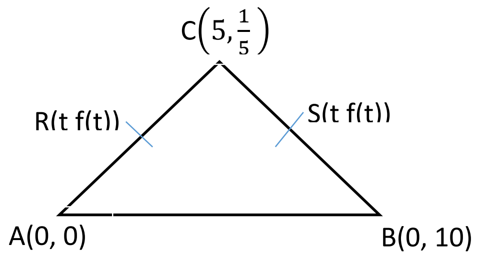
(i) value of k

Area under the curve = 1

$\frac{1}{2}\ x\ 10\ x\ k\ = 1$

$k = \frac{1}{5}$

(ii) equation of the p.d.f



Gradient of $\overline{AC}$ = Gradient of $\overline{AR}$

$\frac{\frac{1}{5}-0}{5-0} = \frac{f(x)-0}{t=0}$

$\frac{1}{25} = \frac{f(x)}{t}$

f(x) $= \frac{1}{25}t$

Gradient of $\overline{BC}$ = Gradient of $\overline{BS}$

$\frac{\frac{1}{5}-0}{5-10} = \frac{f(x)-0}{t=10}$

$-\frac{1}{25} = \frac{f(x)}{t-10}$

f(x) $= \frac{10-t}{25}$

Hence $f(x) = \begin{cases} \frac{1}{25}t, & 0 \le x \le 5 \\ \frac{1}{25}(10-t), & 5 \le x \le 10 \\ 0, & elsewhere \end{cases}$

(iii) E(T): since the graph is symmetrical about t = 5; Hence E(T) = 5

(iv) Probability that a pupil leaves between 4 and 7 minutes after the bell

P(4 < t < 7) = $\frac{1}{25}\int_4^5 tdx + \frac{1}{25}\int_5^7 (10-t)kdx$

$= \frac{1}{25}\left[\frac{t^2}{2}\right]_4^5 + \frac{1}{25}\left[10t - \frac{t^2}{2}\right]_5^7$ = 0.5

Revision exercise 1

1. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx^2 & 0 \le x \le 2 \\ 0, & elsewhere \end{cases}$

(a) Find the value of the constant k (=$\frac{3}{8}$) and sketch f(x)

(b) Find (i) P(X≥1) = $\frac{3}{8}$ (ii) P(0.5≤ x ≤ 1.5) = $\frac{13}{32}$

2. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k & -2 \le x \le 3 \\ 0, & elsewhere \end{cases}$

   (i) Sketch f(x)
   (ii) Find the value of the constant k = $\frac{1}{5}$.
   (iii) Find P(-1.6 ≤ x ≤ 2.1) = 0.74

3. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(4-x) & 1 \le x \le 3 \\ 0, & elsewhere \end{cases}$

   (i) Sketch f(x)
   (ii) Find the value of the constant k = $\frac{1}{4}$.
   (iii) Find P(1.2 ≤ x ≤ 2.4) = 0.66

4. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(x+2)^2 & 0 \le x \le 2 \\ 0, & elsewhere \end{cases}$

   (a) Sketch f(x)
   (b) Find the value of the constant k = $\frac{1}{56}$.
   (c) Find (i) P(0 ≤ x ≤ 1) = $\frac{19}{56}$ (ii) P(X ≥ 1) =$\frac{37}{56}$

5. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(x)^3 & 0 \le x \le c \\ 0, & elsewhere \end{cases}$

   Given that P(X ≤ 0.5) = $\frac{1}{16}$
   (i) Find the value of k and c (k = 1 and k = 4)
   (ii) Sketch f(x)

6. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx & 0 \le x \le 4 \\ 0, & elsewhere \end{cases}$

   (i) Sketch f(x)
   (ii) Find the value of the constant k = $\frac{1}{8}$.
   (iii) Find P(1 ≤ x ≤ 2.5) = 0.328

7. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k, & 0 \le x \le 2 \\ k(2x-3), & 2 \le x \le 3 \\ 0, & elsewhere \end{cases}$

   (i) Sketch f(x)
   (ii) Find the value of the constant k = $\frac{1}{4}$.
   (iii) Find (i) P(X >1) =$\frac{1}{4}$ (ii) P(X > 2.5) = 0.3125 (iii) P(1 ≤ x ≤ 2.3) = 0.3475

8. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} a, & 0 \leq x \leq 1.5 \\ \frac{a}{2}(2-x), & 2 \leq x \leq 3 \\ 0, & elsewhere \end{cases}$

Find (i) value of a= $\frac{16}{25}$ (ii) P(X < 1.6) = 0.9744

Expectation or mean of X

Example 18

A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx^2 & 0 \leq x \leq 3 \\ 0, & elsewhere \end{cases}$

Find the

(i) value of the constant k and sketch f(x)
(ii) the mean, μ
(iii) P(X ≤ μ)

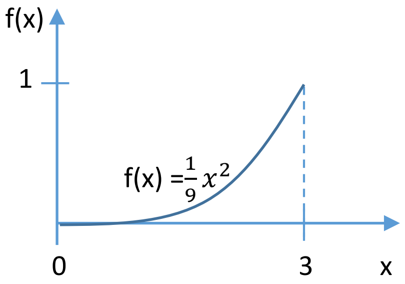
Solution

(i) $\int_0^3 kx^2 dx = 1$

$k\left[\frac{x^3}{3}\right]_0^3 = 1$, k = $\frac{1}{9}$

When x = 0, f(x) =$\frac{1}{9}(0)^2 = 0$

When x = 3, f(x) =$\frac{1}{9}(3)^2 = 1$



(ii) E(X) = $\int_0^3 x.x^2 dx$

=$\frac{1}{9}\left[\frac{x^4}{4}\right]_0^3$ = 2.25

(iii) P(X ≤ μ) =$\frac{1}{9}\int_0^{2.25} x^2 dx$

= $\frac{1}{9}\left[\frac{x^4}{4}\right]_0^{2.25}$

= 0.42

Example 19

A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx^3 & 0 \leq x \leq 2 \\ 0, & elsewhere \end{cases}$

Find (i) the value of the constant k

$\int_0^3 kx^3 dx = 1$

$k\left[\frac{x^4}{4}\right]_0^2 = 1$, k = $\frac{1}{4}$

(ii) mean

E(X) = $\frac{1}{4}\int_0^3 x.x^3 dx = \frac{1}{4}\left[\frac{x^5}{5}\right]_0^2$ = 1.6

(iii) P(X ≤ 1) =$\frac{1}{4}\int_0^1 x^3 dx = \frac{1}{4}\left[\frac{x^4}{4}\right]_0^1 = 0.0625$

Example 20

A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(4x - x^2), & 0 \le x \le 2 \\ 0, & elsewhere \end{cases}$

Find

(i) the value of constant k

$\int_0^2 k(4x - x^2)dx = 1$

$k\left[2x^2 - \frac{x^3}{3}\right]_0^2$, k = $\frac{3}{16}$

(ii) E(X)

$\frac{3}{16}\int_0^2 x(4x - x^2)dx$ = $\frac{3}{16}\left[\frac{4x^3}{3} - \frac{x^4}{4}\right]_0^2$ = 0.25

(iii) P(X ≤ 1) =$\frac{3}{16}\int_0^1 (4x - x^2)dx$ = $\frac{3}{16}\left[2x^2 - \frac{x^3}{3}\right]_0^1$ = 0.3125

Example 21

A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} 3x^k, & 0 \le x \le 1 \\ 0, & elsewhere \end{cases}$

(i) Find the value of k

$3\int_0^1 x^k dx = 1$

$3\left[\frac{x^{k+1}}{k+1}\right]_0^1 = 1$

$3\left[\frac{1^{k+1}}{k+1} - \frac{0^{k+1}}{k+1}\right] = 1$

$\frac{3}{k+1} = 1$

k = 2

(ii) Find the mean

E(X) =$\int_0^1 x(3x^2)dx$ = 3 $\left[\frac{x^4}{4}\right]_0^1$ = 0.75

(iii) Find the value of a such that P(X ≤ a) = 0.5

P(X ≤ a) =3 $\int_0^a x^2 dx$ = 0.5

= 3 $\left[\frac{x^3}{3}\right]_0^a$ = $a^3 - 0^3$ = 0.5

= a^3 = 0.5; a = 0.794

Example 22

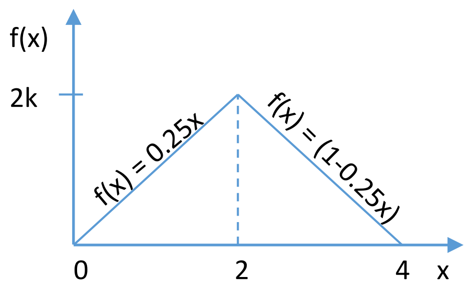
A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{1}{4}x, & 0 \le x \le 2 \\ \left(1 - \frac{1}{4}x\right), & 2 \le x \le 4 \\ 0, & elsewhere \end{cases}$

(i) Sketch f(x)

When x = 0, f(x) = $\frac{1}{4}\ x\ (0) = 0$

When x = 2, f(x) = $\frac{1}{4}\ x\ (2) = 0.25$

When x = 4, f(x) = $\left(1 - \frac{1}{4}(4)\right)$ = 0



(ii) Mean

E(X) = $\frac{1}{4}\int_0^2 x.xdx + \int_2^4 x\left(1-\frac{1}{4}x\right)dx$

$\frac{1}{4}\left[\frac{x^3}{3}\right]_0^2 + \left[\frac{x^2}{2}-\frac{x^3}{12}\right]_2^4$ = 2

(iii) P(X > 3) = $\int_3^4\left(1-\frac{1}{4}x\right)dx$

=$\left[x-\frac{x^2}{8}\right]_3^4$ = 0.125

Example 23

A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(x+2), & -1 \le x \le 0 \\ 2k(1-x), & 0 \le x \le 1 \\ 0, & elsewhere \end{cases}$

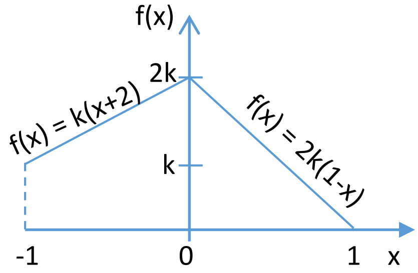
(i) Sketch f(x)

When x = -1, f(x) = k(-1 + 2) = k

When x = 0, f(x) = k(0 + 2) = 2k

When x = 1, f(x) =2k(1-1) = 0

Sketch



(ii) value of k

Area under the graph = 1

$\frac{1}{2}\ x\ 1(k+2k) + \frac{1}{2}\ x\ 1\ x\ 2k = 1$

$k = \frac{2}{5}$

(iii) $P\left(0 < x < 0.5/X > 0\right)$

$P\left(0 < x < 0.5/X > 0\right) = \frac{P(0<x<0.5)}{P(X>0)} = \frac{\frac{4}{5}\int_0^{0.5}(1-x)dx}{\frac{4}{5}\int_0^1(1-x)dx} = \frac{\left[x-\frac{x^2}{2}\right]_0^{0.5}}{\left[x-\frac{x^2}{2}\right]_0^1} = \frac{3/8}{1/2} = 0.75$

(iv) Mean

E(X) = $\frac{2}{5}\int_{-1}^0 x(x+2)dx + \frac{4}{5}\int_0^1 x(1-x)dx$

$=\frac{2}{5}\left[\frac{x^3}{3} + x^2\right]_{-1}^0 + \frac{4}{5}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = -\frac{2}{15}$

Properties of the mean

(i) E(a) = a
(ii) E(ax) = a.E(x)
(iii) E(ax + b) = aE(x) + b
(iv) E(ax - b) = aE(x) – b

Where a and b are constants

Example 24

A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{1}{20}(x+3), & 0 \le x \le 4 \\ 0, & elsewhere \end{cases}$

(i) Sketch f(x)
(ii) Find E(X)

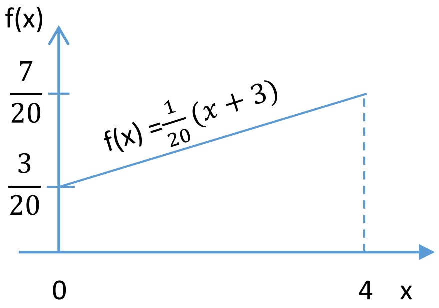
(iii) Find E(2X + 5)

Solution

(i) When x = 0, f(x) = $\frac{1}{20}(0+3) = \frac{3}{20}$

When x = 4, f(x) = $\frac{1}{20}(4+3) = \frac{7}{20}$

Sketch



(ii) E(X) = $\frac{1}{20}\int_0^4 x(x+3)dx$

$= \frac{1}{20}\left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_0^4$

= 2.266

(iii) E(2X + 5) = 2 x 2.266 + 5 = 9.533

Example 25

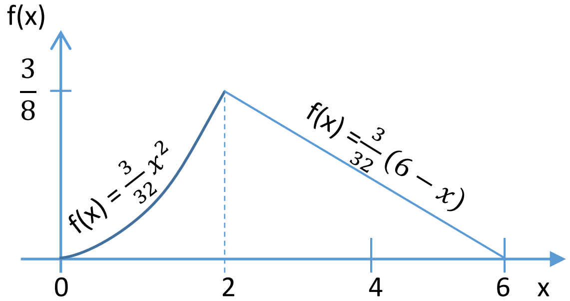
A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{3}{32}x^2, & 0 \le x \le 2 \\ \frac{3}{32}(6-x), & 2 \le x \le 6 \\ 0, & elsewhere \end{cases}$

(i) Sketch f(x)

When x = 0, f(x) = $\frac{3}{32}(0)^2 = 0$

When x = 2, f(x) = $\frac{3}{32}(2)^2 = \frac{3}{8}$

When x =6, f(x) = $\frac{3}{32}(6-6) = 0$



(ii) Find P(X< 4)

P(X<4) = $\frac{3}{32}\int_0^2 x^2dx + \frac{3}{32}\int_2^4 (6-x)dx$

$= \frac{3}{32}\left[\frac{x^3}{3}\right]_0^2 + \frac{3}{32}\left[6x - \frac{x^2}{2}\right]_2^4 = \frac{13}{16}$

(iii) find the mean

E(X) = $\frac{3}{32}\int_0^2 x.x^2dx + \frac{3}{32}\int_2^4 x(6-x)dx$

$= \frac{3}{32}\left[\frac{x^4}{4}\right]_0^2 + \frac{3}{32}\left[3x^2 - \frac{x^3}{3}\right]_2^6$

= 2.875

(iv) Find E(100x -20)

E(100X -20) = 100 x 2.875 – 20 = 267.50

Revision exercise 2

1. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx^2, & 0 \le x \le 4 \\ 0, & elsewhere \end{cases}$

(i) Sketch f(x) (ii) Find E(x) = 3 (iii) find E(2X + 5) = 11

2. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx^2(10-x), & 0 \le x \le 4 \\ 0, & elsewhere \end{cases}$

   (i) Find value of k = $\frac{3}{2500}$ (ii) Find E(x) = 6 (iii) find E(3X - 4) = 14

3. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 5 \le x \le 10 \\ 0, & elsewhere \end{cases}$

   (i) Sketch f(x) (ii) Find value of k = $\frac{2}{75}$ (iii) Find E(x) = $\frac{70}{9}$ (iii) find P(X >8) = 0.48

4. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k[1-(x-2)^2], & 1 \le x \le 3 \\ 0, & elsewhere \end{cases}$

   (i) Find value of k = $\frac{3}{4}$ (ii) sketch f(x) (iii) find E(X) = 2

5. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx(5-x), & 0 \le x \le 5 \\ 0, & elsewhere \end{cases}$

   (i) Find value of k = $\frac{6}{125}$ (ii) sketch f(x) (iii) find E(X) = 2.5

6. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(1-cosx), & 0 \le x \le \pi \\ 0, & elsewhere \end{cases}$

   (i) Find value of k = $\frac{1}{\pi}$ (ii) sketch f(x) (iii) find mean of x = 0.9342

7. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{k}{3}x, & 0 \le x \le 3 \\ k, & 3 \le x \le 4 \\ 0, & elsewhere \end{cases}$

   (i) Sketch f(x) (ii) find k = $\frac{2}{5}$ (iii) find E(X) = 2.6

   (iv) find value of c such that P(X>c) = 0.85; c = 1.5

8. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(x-\frac{1}{a}), & 1 \le x \le 3 \\ 0, & elsewhere \end{cases}$

   Given that P(X> 1) = 0.8,
   Find (i) values of a and k ($\frac{2}{15}, -1$) (ii) probability between 0.5 and 2.5 = 0.6667 (iii) E(X) =1.8

9. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(x+2) & -1 \le x \le 0 \\ 2k, & 0 \le x \le 1 \\ \frac{k}{2}(5-x) & 1 \le x \le 3 \\ 0, & elsewhere \end{cases}$

   (a) Sketch the function f(x)
   (b) Find the value of k (=$\frac{2}{13}$) and the mean (= $\frac{12}{13}$)

10. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} 2kx, & 0 \le x \le 1 \\ k(3-x) & 1 \le x \le 2 \\ 0, & elsewhere \end{cases}$

    (a) Sketch f(x)
    (b) Find the value of k (=$\frac{2}{5}$) and the mean = $\frac{17}{15}$

11. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \alpha(1-cosx), & 0 \le x \le \frac{\pi}{2} \\ \alpha sinx, & \frac{\pi}{2} \le x \le \pi \\ 0, & elsewhere \end{cases}$

    (i) Find value of α (= $\frac{2}{\pi}$) (ii) mean, μ (= 1 +$\frac{\pi}{4}$) (iii) $P\left(\frac{\pi}{3} < x < \frac{3\pi}{4}\right)$ = 0.6982

12. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k_1x, & 1 \le x \le 3 \\ k_2(4-x), & 3 \le x \le 4 \\ 0, & elsewhere \end{cases}$

    (a) Show that $k_2 = 3k_1$
    (b) Find (i) values of k_1 and k_2 (ii) mean, μ

13. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{y+1}{4} & 1 \le y \le k \\ 0, & elsewhere \end{cases}$

Find

(i) Value of k =2

(ii) Expectation Y = 1.6667

(iii) P(1≤ y≤ 1.5) = 0.2813

Solutions to revision exercise 2

8. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(x - \frac{1}{a}), & 1 \le x \le 3 \\ 0, & elsewhere \end{cases}$

Given that P(X> 1) = 0.8,

Find

(i) values of a and k $(\frac{2}{15}, -1)$

$\int_0^3 k\left(x - \frac{1}{a}\right) dx = 1$

$k\left[\frac{x^2}{2} - \frac{x}{a}\right]_0^3 = 1$

$k\left(\frac{9}{2} - \frac{3}{a}\right) = 1$

(9a-6)k = 2a …………… (i)

Given P(X>1) = 0.8

⇨ $\int_1^3 k\left(x - \frac{1}{a}\right) dx = 0.8$

$k\left[\frac{x^2}{2} - \frac{x}{a}\right]_1^3 = 0.8$

$k\left[\left(\frac{9}{2} - \frac{3}{a}\right) - \left(\frac{1}{2} - \frac{1}{a}\right)\right] = 1$

(8a - 4)k = 1.6a ….. (ii)

Eqn.(i) and (ii), a = -1, k = $\frac{2}{15}$

(ii) probability between 0.5 and 2.5

P(0.5 < x < 2.5) =$\frac{2}{15}\int_{0.5}^{2.5}(x+1)dx$

= $\frac{2}{15}\left[\frac{x^2}{2} - \frac{x}{a}\right]_{0.5}^{2.5}$ = 0.6667

(iii) mean

E(X) = $\frac{2}{15}\int_0^3 x(x+1)dx$

=$\frac{2}{15}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^3$ = 1.8

9. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(x+2) & -1 \le x \le 0 \\ 2k, & 0 \le x \le 1 \\ \frac{k}{2}(5-x) & 1 \le x \le 3 \\ 0, & elsewhere \end{cases}$

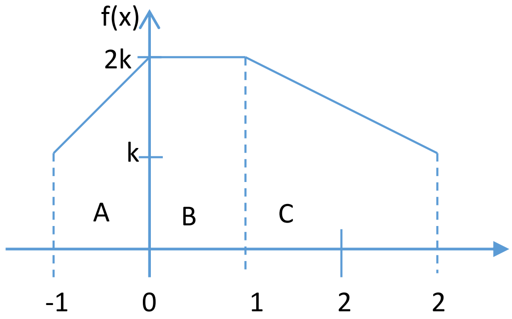
(a) Sketch the function f(x)

For $-1 \le x \le 0$, f(x) = $k(x+2)$

When x = -1, f(x) = k

When x = 0, f(x) =2k

For $0 \le x \le 1$, f(x) = $2k$,
When x = 0, f(x) = 2k
When x = 1, f(x) =2k
For $1 \le x \le 3$, f(x) = $\frac{k}{2}(5-x)$
When x = 1, f(x) = $\frac{k}{2}(5-1)$ =2k
When x = 3, f(x) =$\frac{k}{2}(5-3)$ =k
Sketch



(b)(i) find value of k

Area under the graph = 1

$\frac{1}{2} x\ 1\ (k+2k) + 1\ x\ 2k + \frac{1}{2} x\ 2\ (k+2k)$= 1

k = $\frac{2}{13}$

or

$k\int_{-1}^{0}(x+2)dx + 2k\int_{0}^{1}dx + \frac{k}{2}\int_{1}^{3}(5-x)dx = 1$

$k\left[\frac{x^2}{2}+2x\right]_{-1}^{0} + 2k[x]_0^1 + \frac{k}{2}\left[5x-\frac{x^2}{2}\right]_1^3 = 1$

k = $\frac{2}{13}$

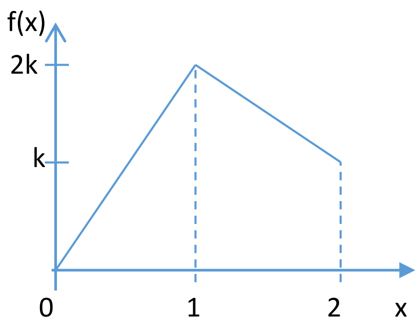
(b) (ii) Find the mean

E(X) =$\frac{2}{13}\int_{-1}^{0}x(x+2)dx + \frac{4}{13}\int_{0}^{1}xdx + \frac{1}{13}\int_{1}^{3}x(5-x)dx$

=$\frac{2}{13}\left[\frac{3}{3}+x^2\right]_{-1}^{0} + \frac{4}{13}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{13}\left[\frac{5x^2}{2}-\frac{x^3}{3}\right]_1^3 = \frac{12}{13}$

10. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} 2kx, & 0 \le x \le 1 \\ k(3-x) & 1 \le x \le 2 \\ 0, & elsewhere \end{cases}$

(a) Sketch f(x)
For $0 \le x \le 1$, f(x) = 2kx
When x = 0, f(x) = 2k(0) = 0
When x = 1, f(x) = 2k(1) =2k
For $1 \le x \le 2$, f(x) = k(3-x)
When x= 1, f(x) = k(3 – 1) = 2k
When x =3, f(x) = k(3 -2) = k
Sketch



(b) Find value of k

Area under the graph = 1

$\frac{1}{2}\ x1\ x\ 2k + \frac{1}{2}\ x\ 1\ (k+2k) = 1$

k = $\frac{2}{5}$

Alternatively

$2k\int_{0}^{1}xdx + k\int_{1}^{2}(3-x)\,dx = 1$

$2k\left[\frac{x^2}{2}\right]_0^1 + k\left[3x-\frac{x^2}{2}\right]_1^2 = 1$

k = $\frac{2}{5}$

(b) Find the mean

E(X) = $\frac{4}{5}\int_0^1 x^2dx + k\int_1^2 x(3-x)\,dx = 1$

= $\frac{4}{5}\left[\frac{x^3}{3}\right]_0^1 + \frac{4}{5}\left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_1^2 = \frac{17}{15}$ =1.133

11. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \alpha(1-cosx), & 0 \le x \le \frac{\pi}{2} \\ \alpha sinx, & \frac{\pi}{2} \le x \le \pi \\ 0, & elsewhere \end{cases}$

(i) Find value of α

$\alpha\int_0^{\frac{\pi}{2}}(1-cosx)dx + \alpha\int_{\frac{\pi}{2}}^{\pi} sinxdx$ = 1

$\alpha[x - sinx]_0^{\frac{\pi}{2}} + \alpha[-cosx]_{\frac{\pi}{2}}^{\pi}$ =1

α = $\frac{2}{\pi}$

(ii) mean, μ

E(X) = $\frac{2}{\pi}\int_0^{\frac{\pi}{2}} x(1-cosx)dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} xsinxdx$

= $\frac{2}{\pi}\int_0^{\frac{\pi}{2}}(x - xcosx)dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} xsinxdx$

= $\frac{2}{\pi}\left[\left[\frac{x^2}{2}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} xcosxdx\right] + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} xsinxdx$

=$\frac{2}{\pi}\left[\left[\frac{x^2}{2}\right]_0^{\frac{\pi}{2}} - [xsinx + \cos x]_0^{\frac{\pi}{2}}\right] + \frac{2}{\pi}[-xcosx + sinx]_{\frac{\pi}{2}}^{\pi}$

=$\frac{2}{\pi}\left[\frac{x^2}{2} - (xsinx + cosx)\right]_0^{\frac{\pi}{2}} = 1 + \frac{\pi}{4}$

(iii) $P\left(\frac{\pi}{3} < x < \frac{3\pi}{4}\right)$

$P\left(\frac{\pi}{3} < x < \frac{3\pi}{4}\right) = \frac{2}{\pi}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}(1-cosx)dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} sinxdx$ = 1

$\alpha[x - sinx]_{\frac{\pi}{3}}^{\frac{\pi}{2}} + \alpha[-cosx]_{\frac{\pi}{2}}^{\frac{3\pi}{4}}$ = 0.6982

12. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k_1x, & 1 \le x \le 3 \\ k_2(4-x), & 3 \le x \le 4 \\ 0, & elsewhere \end{cases}$

(a) Show that $k_2 = 3k_1$

For$1 \le x \le 3$, f(x) = $k_1(x)$

f(3) = $3k_1$ ....(i)

For$3 \le x \le 4$, f(x) =$k_2(4-x)$

f(3) = k_2

Eqn. (i) and eqn. (ii)

$k_2 = 3k_1$

(b) Find (i) values of k_1 and k_2

$k_1\int_1^3 xdx + 3k_1\int_3^4(4-x)dx = 1$

$k_1\left[\frac{x^2}{2}\right]_1^3 + 3k_1\left[4x - \frac{x^2}{2}\right]_3^4 = 1$

$k_1 = \frac{2}{11}$

$k_2 = \frac{6}{11}$

(c) mean, μ

$E(X) = \frac{2}{11}\int_1^3 x^2 dx + \frac{6}{11}\int_3^4 x(4-x)dx$

$\frac{2}{11}\left[\frac{x^3}{3}\right]_1^3 + 3k_1\left[2x^2 - \frac{x^3}{3}\right]_3^4 = 2.485$

13. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{y+1}{4} & 1 \le y \le k \\ 0, & elsewhere \end{cases}$

Find

(a) The value of k (06marks)

$\int_0^k \frac{(y+1)}{4} dy = \frac{1}{4}\left[\frac{y^2}{2} + y\right]_0^k = 1$

$\frac{1}{4}\left[\left(\frac{k^2}{2} + k\right) - 0\right] = 1$

$k^2 + 2k - 8 = 0$

(k + 4)(k-2) = 0

Either

k + 4 = 0; k = -4

Or

k-2 = 0; k = 2

$\therefore k = 2$ (since k is greater than zero)

(b) The expectation of Y (03marks)

E(Y) = $\int_0^2 y dy$

$= \int_0^2 y\left[\frac{y+1}{4}\right] dy$

$= \int_0^2 \left(\frac{y^2+y}{4}\right) dy$

$= \frac{1}{4}\left[\frac{y^3}{3} - \frac{y^2}{2}\right]_0^2$

$= \frac{1}{4}\left[\left(\frac{8}{3} - \frac{4}{2}\right) - 0\right] = \frac{7}{6} = 1.166$

(c) $P(1 \le Y \le 1.5)$ (03marks)

$P(1 \le Y \le 1.5) = \int_1^{1.5}\left[\frac{y+1}{4}\right] dy$

$= \frac{1}{4}\left[\frac{y^2}{2} + y\right]_1^{1.5}$

$= \frac{1}{4}\left[\left(\frac{(1.5)^2}{2} + 1.5\right) - \left(\frac{1}{2} + 1\right)\right]$

$= \frac{1}{4}(2.625 - 1.5)$

= 0.28125

Variance of X

For a continuous random variable with p.d.f, f(x)

Var(X) = $EX^2 - [E(X)]^2$ or Var(X) = $E(X^2) - \mu^2$

Where $E(X^2) = \int x^2(x)dx$ and μ = mean

Properties of variance

(i) Var(a) = 0
(ii) Var(ax) = a^2Var(x)
(iii) Var(ax + b) = a^2Var(x)
(iv) Var(ax – b) = a^2Var(X)

Where a and b are constants

Example 26

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} k(1-x^2), & 0 \le x \le 1 \\ 0, & elsewhere \end{cases}$

Find

(i) the value of k

$k\int_0^1 (1-x^2)\,dx = 1$

$k\left[x - \frac{x^3}{3}\right]_0^1 = 1$

k = 1.5

(ii) E(X)

E(X) =$1.5\int_0^1 x(1-x^2)\,dx$

=$1.5\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{3}{8}$

(iii) Var(X)

E(X2) = $1.5\int_0^1 x^2(1-x^2)\,dx$

= $1.5\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{1}{5}$

Var(X) = $EX^2 - [E(X)]^2$

=$\frac{1}{5} - \left(\frac{3}{8}\right)^2 = \frac{19}{320}$

Example 27

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{1}{8}x, & 0 \le x \le 4 \\ 0, & elsewhere \end{cases}$

Find

(i) E(X)

E(X) =$\frac{1}{8}\int_0^4 x.x\,dx = \frac{1}{8}\left[\frac{x^3}{3}\right]_0^4$ = 2.667

(ii) Var(X)

E(X^2) = $\frac{1}{8}\int_0^4 x^2.x\,dx = \frac{1}{8}\left[\frac{x^4}{4}\right]_0^4$= 8

Var(X) = $EX^2 - [E(X)]^2$

= 8 – $(2.667)^2$ = 0.887

(iii) Standard deviation

s.d = $\sqrt{Var(X)}$

=$\sqrt{0.887}$ = 0.942

(iv) Var(3x + 2) = 0.887 x 3 = 7.983

Example 28

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{4}{25}(5-2x), & 0 \le x \le 2.5 \\ 0, & elsewhere \end{cases}$

Find

(i) Mean

E(X) $=\frac{4}{25}\int_0^{2.5} x(5-2x)\,dx = \frac{4}{25}\left[\frac{5x^2}{2}-\frac{2x^3}{3}\right]_0^{2.5}$ = 0.833

(ii) Standard deviation

E(X^2) $=\frac{4}{25}\int_0^{2.5} x^2(5-2x)\,dx = \frac{4}{25}\left[\frac{5x^3}{3}-\frac{2x^4}{4}\right]_0^{2.5}$ = 1.041

Var(X) = EX^2 – [E(X)]^2 = 1.041 – (0.5625)^2 = 0.347

s.d = $\sqrt{Var(X)}$= $\sqrt{0.347}$ = 0.59

Example 29

A random variable x of a continuous p.d.f is given by $f(x)=\begin{cases}\frac{3}{4}(1+x^2), & 0\le x\le 1\\ 0, & elsewhere\end{cases}$

Find

(i) Mean

E(X) $=\frac{3}{4}\int_0^1 x(1+x^2)\,dx = \frac{3}{4}\left[\frac{x^2}{2}+\frac{x^4}{4}\right]_0^1$ = 0.5625

(ii) Standard deviation

E(X^2) $=\frac{3}{4}\int_0^1 x^2(1+x^2)\,dx = \frac{3}{4}\left[\frac{x^3}{3}+\frac{x^5}{5}\right]_0^1$ = 0.4

Var(X) = 0.4 – (0.525)2 = 0.835

s.d = $\sqrt{0.0835}$ = 0.289

(iii) P($|X-\mu|<\sigma$)

P($|X-\mu|<\sigma$) = P($|X-0.5625$ |< x < 0.289)

= P(0.2735 < x < 0.8515

$\frac{3}{4}\int_{0.2735}^{0.8515}(1+x^2)\,dx = \frac{3}{4}\left[x+\frac{x^3}{3}\right]_{0.2735}^{0.8515}$ = 0.583

Revision exercise 3

1. A random variable x of a continuous p.d.f is given by $f(x)=\begin{cases}kx^2, & 0\le x\le 4\\ 0, & elsewhere\end{cases}$
   (a) Sketch f(x)
   (b) Find (i) value of k (=$\frac{3}{64}$) (ii) E(X) = 3 and var (X) = 0.6 (iii) P(1<X<2) = $\frac{7}{64}$
2. A random variable x of a continuous p.d.f is given by $f(x)=\begin{cases}kx, & 0\le x\le 1\\ k(2-x) & 0\le x\le 1\\ 0, & elsewhere\end{cases}$

   Find (i) constant k = 1 (ii) E(X) = 1 (iii) var(X) =$\frac{1}{6}$ (iv)P(0.75 < X < 1.5) = $\frac{19}{32}$ (v) mode =1
3. A random variable x of a continuous p.d.f is given by $f(x)=\begin{cases}\frac{1}{27}x^2, & 0\le x\le 3\\ \frac{1}{3} & 3\le x\le 5\\ 0, & elsewhere\end{cases}$
   (a) Sketch f(x)
   (b) Find (i) E(X) = 3417 (ii) standard deviation = 1.008
4. A random variable x of a continuous p.d.f is given by $f(x)=\begin{cases}\frac{k}{x(4-x)}, & 1\le x\le 3\\ 0, & elsewhere\end{cases}$

(i) Show that k = $\frac{3}{Inx}$

(ii) Find (i) E(X) = 2 (ii) Var(X) = 4-$\frac{4}{Inx}$

5. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} k(ax - x^2), & 0 \le x \le 2 \\ 0, & elsewhere \end{cases}$

(i) Show that k = $\frac{8}{6a-8}$

(ii) Given that E(X) = 1, find the values of a (=2) and k(=0.75)

(iii) For the above values of a and k, find Var(X) = 0.2

6. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} 12(x^2 - x^3), & 0 \le x \le 1 \\ 0, & elsewhere \end{cases}$

Find the (i) mean = 0.6 (ii) standard deviation = 0.2

7. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{k}{\beta}, & 0 \le x \le \beta \\ 0, & elsewhere \end{cases}$

Find (i) value of k (=1) (ii) mean = $\frac{\beta}{2}$ (iii) standard deviation = $\frac{\beta}{2\sqrt{3}}$

8. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{1}{8}(x + 1), & 2 \le x \le 4 \\ 0, & elsewhere \end{cases}$

Find (i) mean = $\frac{37}{12}$ (ii) var(X) = $\frac{47}{144}$ (iii) P(2.5 < x< 3) = 0.234

9. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} k(1 - x)^2, & 2 \le x \le 4 \\ 0, & elsewhere \end{cases}$

Find (i) constant k = $\frac{3}{26}$ (ii) mean = $\frac{1}{4}$ (iii) standard deviation = 0.94

10. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 0 \le x \le 2 \\ k(4 - x) & 2 \le x \le 4 \\ 0, & elsewhere \end{cases}$

Find (i) value of k = $\frac{1}{4}$ (ii) E(X) = 2 (iii) Var(X) = $\frac{2}{3}$ (iv) P(X< 1) = $\frac{1}{8}$ (iv) P(X<X<3) = $\frac{3}{8}$

Mode

This is the value of f(x) is maximum in the given range of x.

(i) The mode is obtained from $\frac{d}{dx}(fx) = 0$

The maximum value is confirmed if $\frac{d^2}{dx^2}(fx) =$ negative

(ii) When a sketch of f(x) is drawn, the value of x for which f(x) is maximum gives the mode.

Note: for any line the mode can be determined from a sketch of f(x)

Example 30

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} k(2 + x)(4 - x), & 0 \le x \le 4 \\ 0, & elsewhere \end{cases}$

Find

(i) Value of k

$k \int_0^4 (2 + x)(4 - x)dx = 1$

$k \int_0^4 (8 + 2x - x^2)dx = 1$

$\left[8x + x - \frac{x^3}{3}\right]_0^4 = 1;$ k $=\frac{3}{80}$

(ii) Mode

$\frac{d}{dx}(fx) = 0$

$\frac{d}{dx}\frac{3}{80}(8 + 2x - x^2) = 0$

$\frac{3}{80}(2 - 2x) = 0$; x = 1

∴mode = 1

Example 30

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{1}{108}x(6-x)^2, & 0 \le x \le 6 \\ 0, & elsewhere \end{cases}$

Find

(i) Mean

E(X) = $\int_0^6 \frac{1}{108}x^2(6-x)^2dx$

$=\frac{1}{108}\int_0^6(36x^2 - 12x^3 + x^4)dx$

$=\frac{1}{108}\left[12x^3 - 3x^4 + \frac{x^5}{5}\right]_0^6$ = 2.4

(ii) Standard deviation

E(X²) = $\int_0^6 \frac{1}{108}x^3(6-x)^2dx$

$=\frac{1}{108}\int_0^6 \frac{1}{108}(36x^3 - 12x^4 + x^5)dx$

$=\frac{1}{108}\left[9x^4 - \frac{12x^5}{5} + \frac{x^6}{6}\right]_0^6$ = 7.2

s.d = $\sqrt{7.2 - (2.4)^2}$ = 1.2

(iii) mode

$\frac{d}{dx}(fx) = 0$

$\frac{d}{dx}\frac{1}{108}x(6-x)^2 = 0$

$\frac{d}{dx}\frac{1}{108}(36x - 12x^2 + x^3)$=0

$\frac{1}{108}(36 - 24x + 3x^2)$ = 0

$(6-x)(2-x) = 0$

x = 6 or x = 2

∴ mode = 2 or 6

Example 31

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} ksinx, & 0 \le x \le \pi \\ 0, & elsewhere \end{cases}$

Find

(i) value k

$\int_0^\pi ksinx\ dx = 1$

$k[-cosx]_0^\pi = 1$

k[-cosπ- -cos0) = 1

k = $\frac{1}{2}$

(ii) P(X ≥ $\frac{\pi}{3}$)

(iii) P$\left(\ge \frac{\pi}{3}\right)$ =$\frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} sinx\ dx$ =$k[-cosx]_{\frac{\pi}{3}}^{\pi} = \frac{3}{4}$

(iv) Mean

E(x) = $\frac{1}{2}\int_0^\pi xsinx\ dx$

Sign	derivative	integral sign
+	x	sinx
-	1	-cosx
+	0	-sinx

⇨ E(x) = $\frac{1}{2}\int_0^\pi xsinx\ dx$

=$\frac{1}{2}[-xcosx + sinx]_0^\pi$

= $\frac{\pi}{2}$

(v) Var (X)

$E(X^2) = \frac{1}{2}\int_0^{\pi} x^2 sinx\ dx$

Sign	Derivative	Integral sign
+	x^2	sinx
-	2x	-cosx
+	2	-sinx
-	0	cosx

⇨ $E(X^2) = \frac{1}{2}\int_0^{\pi} x^2 sinx\ dx = \frac{1}{2}[-x^2cosx + 2xsinx + 2cosx]_0^{\pi} = \frac{\pi^2-4}{2}$

∴$Var(X) = \frac{\pi^2-4}{2}$

(vi) Mode

$\frac{d}{dx}\left(\frac{1}{2}sinx\right) = 0$

$\frac{1}{2}cosx = 0$

$x = 90^0$

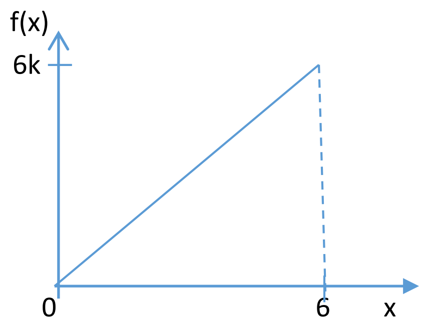
∴mode = $\frac{\pi}{2}$

Example 32

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} kx\ , & 0 \le x \le 6 \\ 0, & elsewhere \end{cases}$

(a) Sketch f(x)

When x= 0, f(x) = k(0) = 0
When x= 6, f(x) = k(6) = 6k



(b) value of k

Area under the graph = 1

$\frac{1}{2}\ x\ k\ x\ 6\ x\ 6 = 1$

k = $\frac{1}{18}$

(c) mode = 6

Median

This is the value of f(x) for which $\int_a^m f(x) = 0.5$; where m is the median, and a is the lower limit.

Example 33

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{1}{8}x\ , & 0 \le x \le 4 \\ 0, & elsewhere \end{cases}$

Find the median

$\int_0^m \frac{1}{8}xdx = 0.5$

$\left[\frac{1}{16}x^2\right]_0^m = 0.5$

$\frac{m^2}{16} = 0.5$; m = $\sqrt{8} = \pm 2.828$

Median = 2.828 (since it falls in the range)

Example 34

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{2}{5}(x+2), & -1 \le x \le 0 \\ \frac{4}{5}(1-x) & 0 \le x \le 10 \\ 0, & elsewhere \end{cases}$

Find the median

Solution

We need to first integrate the first interval to check if it is ≥ 0.5. if not the median lies in the second interval

$\int_{-1}^{0} \frac{2}{5}(x+2)dx = \frac{2}{5}\left[\frac{x^2}{2} + 2x\right]_{-1}^{0} = 0.6$

It shows that the median lies in the first interval

Then $\int_{-1}^{m} \frac{2}{5}(x+2)dx = \frac{2}{5}\left[\frac{x^2}{2} + 2x\right]_{-1}^{m} = 0.5$

m = -0.129 or m = -3.871

the median = -0.129 since it lies in the range

Example 34

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{2}{3}(x+1), & -1 \le x \le 0 \\ \frac{1}{3}(2-x) & 0 \le x \le 2 \\ 0, & elsewhere \end{cases}$

Find the median

We need to first integrate the first interval to check if it is ≥ 0.5. if not the median lies in the second interval

$\int_{-1}^{0} \frac{2}{3}(x+1)dx = \frac{2}{5}\left[\frac{x^2}{2} + x\right]_{-1}^{0} = \frac{1}{3}$

It shows that the median lies in the second interval

Then $\frac{1}{3} + \frac{1}{3}\int_{0}^{m}(2-x)dx = \frac{1}{2}$

$\frac{1}{3}\left[2x - \frac{x^2}{2}\right]_0^m = \frac{1}{6}$; m =0.268

Revision exercise 4

1. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} kx(4-x^2), & 0 \le x \le 2 \\ 0, & elsewhere \end{cases}$

   Find
   (i) value of the constant =0.25 (iii) mean = 1.067
   (ii) median x =2.613 (iv) standard deviation = 0.442

2. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 0 \le x \le 1 \\ k(2-x) & 1 \le x \le 2 \\ 0, & elsewhere \end{cases}$

   Find
   (i) constant k = 1 (ii) median = 1 (iii) mode = 1

3. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} kx(4-x^2), & 0 \le x \le 2 \\ 0, & elsewhere \end{cases}$

   Find
   (i) value of the constant =$\frac{1}{4}$ (iii) mean = 1.0667
   (ii) median x =2.6131 (iv) standard deviation = 0.4422

4. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \alpha, & 2 \le x \le 3 \\ \alpha(x-2) & 3 \le x \le 4 \\ 0, & elsewhere \end{cases}$

   (a) sketch f(x)
   (b) find (i) constant α = 0.4 (ii) median, m = 3.225 (iii) P(2.5 < x < 3.5) =0.65

5. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \beta, & 0 \le x \le 2 \\ \beta(3-x) & 2 \le x \le 3 \\ 0, & elsewhere \end{cases}$

   Find (i) value of β =0.4 (ii) mean = $\frac{19}{15}$ (iii) standard deviation = $\frac{5}{4}$ (iv) P(X<μ –σ) =0.207

6. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ \frac{1}{2} & 1 \le x \le k \\ 0, & elsewhere \end{cases}$

   (i) Sketch f(x)
   (ii) Find (i) value of k = $\frac{7}{3}$ (ii) mean = $\frac{49}{36}$ (iii) median = $\frac{4}{3}$

7. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 0 \le x \le 1 \\ k & 1 \le x \le 2 \\ 0, & elsewhere \end{cases}$

   (i) Sketch f(x)
   (ii) Find (i) value of k = $\frac{2}{3}$ (ii) mean = $\frac{49}{36}$ (iii) median = 1.25 (iv) P($|X-m|$>0.5) = $\frac{17}{48}$

8. A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} 2k(x+1), & -1 \le x \le 0 \\ k(2-x) & 0 \le x \le 2 \\ 0, & elsewhere \end{cases}$

   (i) Sketch f(x)
   (ii) Find (i) value of k = $\frac{1}{3}$ (ii) mean = $\frac{1}{3}$ (iii) Var(X) = $\frac{5}{18}$ (iv) mode =0

Cumulative distribution function, F(x)

The cumulative distribution function F(x) is defined by F(x) = $\int_a^x fx)dx$

Steps in finding F(x)

- For each interval, integrate its function from lower limit to x with respect to x.
- Substitute the upper limit in the integral and carry it forward to the next interval
- Continue the process until when the last upper limit has been substituted to get a 1.

Example 35

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{1}{6}(x+1), & 1 \leq x \leq 3 \\ 0, & elsewhere \end{cases}$

Find F(x)

Solution

F(x) = $\frac{1}{6}\int_1^x (x+1)dx = \frac{1}{6}\left[\frac{x^2}{2}+x\right]_1^x = \frac{1}{6}\left\{\left(\frac{x^2}{2}+x\right)-\left(\frac{1^2}{2}+1\right)\right\}$

F(x) = $\frac{1}{6}\left(\frac{x^2}{2}+x-\frac{3}{2}\right)$

F(3) =$\frac{1}{6}\left(\frac{3^2}{2}+3-\frac{3}{2}\right)$ =1

∴ F(x) = $\begin{cases} 0 & x \leq 1 \\ \frac{1}{6}\left(\frac{x^2}{2}+x-\frac{3}{2}\right), & 1 \leq x \leq 3 \\ 1, & x \geq 3 \end{cases}$

Example 36

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{3}{26}(1-x)^2, & 2 \leq x \leq 4 \\ 0, & elsewhere \end{cases}$

Find F(x)

F(x) = $\frac{3}{26}\int_2^x (1-x)^2 dx$= $\frac{3}{26}\int_2^x (1-2x+x^2)dx$ = $\frac{3}{26}\left[x-x^2+\frac{x^3}{3}\right]_2^x$

= $\frac{3}{26}\left\{\left(x-x^2+\frac{x^3}{3}\right)-\left(2-2^2+\frac{2^3}{3}\right)\right\}$ =$\frac{3}{26}\left(x-x^2+\frac{x^3}{3}-\frac{2}{3}\right)$

F(4) =$\left(4-4^2+\frac{4^3}{3}-\frac{2}{3}\right)$=1

∴ F(x) = $\begin{cases} 0 & x \leq 2 \\ \frac{3}{26}\left(x-x^2+\frac{x^3}{3}-\frac{2}{3}\right), & 2 \leq x \leq 3 \\ 1, & x \geq 4 \end{cases}$

Example 37

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ (2-x), & 1 \le x \le 2 \\ 0, & elsewhere \end{cases}$

Find F(x)

For $0 \le x \le 1$, F(x) = $\int_o^x x dx$ = $\left[\frac{x^2}{2}\right]_0^x = \left(\frac{x^2}{2} - \frac{0^2}{2}\right) = \frac{x^2}{2}$

F(1) = $\frac{1^2}{2} = \frac{1}{2}$

For $1 \le x \le 2$; F(x) = $\frac{1}{2} + \int_1^x (2-x)dx = \frac{1}{2} + \left[2x - \frac{x^2}{2}\right]_1^x$ =$\frac{1}{2} + \left\{\left(2x - \frac{x^2}{2}\right) - \left(2 - \frac{1^2}{2}\right)\right\}$

=$\left(2x - \frac{x^2}{2}\right) - 1$

F(x) = $\left(2x2 - \frac{2^2}{2}\right) - 1$= 1

$$\therefore F(x) = \begin{cases} 0 & x \le 1 \\ \frac{x^2}{2}, & 0 \le x \le 1 \\ \left(2x - \frac{x^2}{2}\right) - 1, & 1 \le x \le 2 \\ 1, & x \ge 2 \end{cases}$$

Example 38

A random variable x of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{2}{5}, & 0 \le x \le 2 \\ \frac{2}{5}(3-x), & 2 \le x \le 3 \\ 0, & elsewhere \end{cases}$

Find F(x)

For $0 \le x \le 2$, F(x) =$\frac{2}{5}\int_0^x dx = \frac{2}{5}[x]_0^x = \frac{2}{5}\{x - 0\} = \frac{2}{5}x$

F(2) = $\frac{2}{5}x2 = \frac{4}{5}$

For $2 \le x \le 3$, F(X) =$\frac{4}{5} + \frac{2}{5}\int_0^x (3-x)dx = \frac{4}{5} + \frac{2}{5}\left[3x - \frac{x^2}{2}\right]_2^x$=$\frac{4}{5} + \frac{2}{5}\left(3x - \frac{x^2}{2}\right) - \left(3x2 - \frac{2^2}{2}\right)$

F(x) =$\frac{2}{5}\left(3x - \frac{x^2}{2}\right) - \frac{4}{5}$

F(3) =$\frac{2}{5}\left(3x3 - \frac{3^2}{2}\right) - \frac{4}{5}$ = 1

$$\therefore F(x) = \begin{cases} 0 & x \le 0 \\ \frac{2}{5}x, & 0 \le x \le 2 \\ \frac{2}{5}\left(3x - \frac{x^2}{2}\right) - \frac{4}{5}, & 2 \le x \le 3 \\ 1, & x \ge 3 \end{cases}$$

Finding the median, quartiles and probability from F(x)

- The median is the value of m for which F(m) = 0.5
- The lower quartile is the value q_1 for which $F(q_1) = 0.25$
- The upper quartile is the value q_3 for which $F(q_3) = 0.75$

Example 39

The continuous random variable X has a cumulative distribution function given below

$$F(x) = \begin{cases} 0 & x \le 0 \\ \frac{x^2}{16} & 0 \le x \le 4 \\ 1 & x \ge 4 \end{cases}$$

Find

(i) P(0.3≤ X ≤ 1.8)

P(0.3≤ X ≤ 1.8) = F(1.8) – F(0.3) = $\frac{1.8^2}{16} - \frac{0.3^2}{16}$ = 0.197

(ii) Median, m

F(m) = 0.5

$\frac{m^2}{16} = 0.5; m = \pm 2.828$

$median = 2.828\ (since\ it\ is\ within\ the\ range$

(iii) Interquartile range

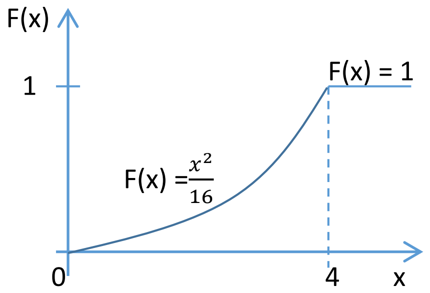
$F(q_1) = 0.25$

$\frac{q_1^2}{16} = 0.25;\ q_1 = 2$

$F(q_3) = 0.75$

$\frac{q_3^2}{16} = 0.75;\ q_3 = 3.464$

Interquartile range = 3.464 – 2 = 1.464



Example 40

The continuous random variable X has a c.d.f given by $F(x) = \begin{cases} 0 & x \le 0 \\ 2x - x^2 & 0 \le x \le 1 \\ 1 & x \ge 4 \end{cases}$

Find

(i) F(X≤0.5)

F(X≤0.5) = F(0.5) – F(0) =$(2(0.5)-(0.5)^2)-(2(0)-(0)^2)$=0.75

(ii) Median, m

F(m) = 0.5

$(2(m)-(m)^2)=0.5$

$m^2-2m+0.5=0$

m = 1.71 or m = 0.293

m = 0.293 (since it is in the range)

(iii) Interquartile range

F(q_1) = 0.25

$2q_1-q_1^2=0.25;\ q_1=0.134$

F(q_3) = 0.75

$2q_3-q_3^2=0.75;\ q_3=0.5$

Interquartile range = 0.5 – 0.134 = 0.366

Example 40

The cumulative distribution function is given by $F(x)=\begin{cases}0 & x\le 0\\ \frac{x^2}{6} & 0\le x\le 2\\ -\frac{x^2}{3}+2x-2 & 2\le x\le 3\\ 1 & x\ge 3\end{cases}$

Find

(i) P(1 ≤ x ≤ 2.5)

P(1 ≤ x ≤ 2.5) = P(2.5) – P(1)

$-\frac{2.5^2}{3}+2x\ 2.5-2-\frac{1^2}{6}$ =0.75

(ii) Median, m

P(0 ≤ x ≤ 2) = F(2) –F(0)

$=\frac{2^2}{6}-\frac{0^2}{6}=\frac{2}{3}$

Since $\frac{2}{3}>0.5$ the median lies between $0\le x\le 2$

F(m) =0.5

$\frac{m^2}{6}=0.5$

$m=\pm 1.73$

Median = 1.73

Revision exercise 5

1. The random variable X has a probability density function f(x) =$\begin{cases}\frac{3}{8}x^2 & 0\le x\le 2\\ 0 & elsewhere\end{cases}$

   Find

   (i) Sketch F(X)

   (ii) Cumulative distribution function; F(x) =$\begin{cases}0 & x<2\\ \frac{1}{8}x^3 & 0\le x\le 2\\ 1 & x\ge 0\end{cases}$

   (iii) Median, m = 1.59

2. The random variable X has a probability density function f(x) = $\begin{cases} \frac{1}{4}(4-x) & 1 \le x \le 3 \\ 0 & elsewhere \end{cases}$

Find

(i) cumulative mass function; F(x) = $\begin{cases} 0 & x \le 1 \\ \frac{1}{8}(8x - x^2 - 7) & 1 \le x \le 3 \\ 1 & x \ge 3 \end{cases}$

(ii) P(1.5≤x≤2)=$\frac{9}{32}$

(iii) median, m = 1.764

(iv) sketch F(x)

3. The random variable X has a probability density function f(x) = $\begin{cases} k & 1 \le x \le 6 \\ 0 & elsewhere \end{cases}$

Find

(i) Value of k = $\frac{1}{5}$

(ii) Cumulative function, F(x) = $\begin{cases} 0 & x \le 1 \\ \frac{1}{5}(x-1) & 1 \le x \le 6 \\ 1 & x \ge 6 \end{cases}$

(iii) Interquartile range =2.5

4. The random variable X has probability density function f(x) = $\begin{cases} \frac{1}{4} & 0 \le x \le 2 \\ \frac{1}{4}(2x-3) & 2 \le x \le 3 \\ 0 & elsewhere \end{cases}$

Find

(i) Cumulative function, F(x) = $\begin{cases} 0 & x \le 1 \\ \frac{x}{4} & 0 \le x \le 2 \\ \frac{1}{4}(x^2 - 3x + 4) & 2 \le x \le 3 \\ 1 & x \ge 3 \end{cases}$

(ii) Median, m= 2

(iii) Sketch F(x)

5. The random variable X has a cumulative distribution function, F(x) = $\begin{cases} 0 & x \le 0 \\ x^4 & 1 \le x \le 1 \\ 1 & x \ge 1 \end{cases}$

Find

(i) P(0.3 ≤ x ≤ 0.6) =0.1215

(ii) Median, m = 0.841

(iii) The value of a such that P(X>a) = 0.88

6. The random variable X has a probability density function f(x) = $\begin{cases} \frac{1}{3} & 0 \le x \le 3 \\ 0 & elsewhere \end{cases}$

Find (i) E(x) = 1.5 (ii) Var (X) = 0.75 (iii) P(X> 1.8) =0.4 (iv) P(1.1 < x <1.7) =0.2

(v) cumulative distribution function, F(x) = $\begin{cases} 0 & x \le 0 \\ \frac{1}{3}x & 0 \le x \le 3 \\ 1 & x \ge 3 \end{cases}$

7. The random variable X has a probability density function f(x) = $\begin{cases} kx^2 & 1 \le x \le 2 \\ 0 & elsewhere \end{cases}$

Find

(i) Value of k = $\frac{3}{7}$ (ii) standard deviation = 0.272 (iii) median, m = 1.65

(ii) Cumulative mass function, F(x) = $\begin{cases} 0 & x \leq 1 \\ \frac{1}{7}(x^3 - 1) & 1 \leq x \leq 2 \\ 1 & x \geq 2 \end{cases}$

8. The continuous random variable X has a p.d.f given by f(x) = $\begin{cases} k(4 - x^2) & 0 \leq x \leq 2 \\ 0 & elsewhere \end{cases}$

Find (i) constant k (= $\frac{3}{16}$) (ii) E(x) = $\frac{3}{4}$ (iii) Var(X) = $\frac{19}{80}$ (iv) median = 0.695

(v) cumulative distribution function, F(X) = $\begin{cases} 0 & x \leq 0 \\ \frac{3}{4}x - \frac{1}{16}x^3 & 0 \leq x \leq 2 \\ 1 & x \geq 2 \end{cases}$

(vi) = P(0.69 ≤ x ≤ 0.7)= 0.007

9. The continuous random variable X has a p.d.f given by f(x) = $\begin{cases} \frac{1+x}{6} & 1 \leq x \leq 3 \\ 0 & elsewhere \end{cases}$

(i) Sketch f(x)
(ii) Find the mean = $\frac{19}{9}$
(iii) Find m such that P(X≤ m) 0.5; m = 2.16
(iv) Determine cumulative function, F(X) and sketch it

F(X) = $\begin{cases} 0 & x \leq 0 \\ \frac{1}{5}x + \frac{1}{12}x^2 - \frac{1}{4} & 1 \leq x \leq 3 \\ 1 & x \geq 3 \end{cases}$

10. A factory is supplied with flour at the beginning of each week. The weekly demand, X thousand tones for flour from this factory is a continuous random variable having a probability density function f(x) = $\begin{cases} k & 1 \leq x \leq 3 \\ 0 & elsewhere \end{cases}$

Find
(i) Value of k = 5
(ii) Mean of x = $\frac{1}{6}$
(iii) Variance of x = $\frac{5}{252}$

11. The continuous random variable X has a p.d.f given by f(x) = $\begin{cases} \frac{1}{4} & 0 \leq x \leq 1 \\ \frac{x^3}{5} & 1 \leq x \leq 2 \\ 0 & elsewhere \end{cases}$

Find

(i) Cumulative mass function, F(x) and sketch it F(x) = $\begin{cases} 0 & x < 0 \\ \frac{1}{4}x & 0 \leq x \leq 1 \\ \frac{1}{5} + \frac{x^4}{20} & 1 \leq x \leq 2 \\ 1 & x \geq 2 \end{cases}$

(ii) Median, m=1.565 (iii) interquartile range = 0.821

12. The continuous random variable X has a p.d.f given by f(x) = $\begin{cases} k(x+3) & -3 \leq x \leq 3 \\ 0 & elsewhere \end{cases}$

(a) Show that k = $\frac{1}{18}$
(b) Find (i) E(x) =1, (ii) Var (x) = 2 (iii) Lower quartile, $q_1 = 0$
(c) Given that E(ax + b) = 0 and Var(ax+b) = 1, find the values of a and b where a>0 (a = b) = $\frac{1}{\sqrt{2}}$

13. The continuous random variable X has a p.d.f given by f(x) = $\begin{cases} kx & 0 \leq x \leq 8 \\ 8k & 8 \leq x \leq 9 \\ 0 & elsewhere \end{cases}$

(a) Sketch f(x)

(b) Find value of k = 0.025 (ii) P(X>6) =0.55

(c) Find F(X) ==$\begin{cases} 0 & x < 0 \\ 0.0125x & 0 \le x \le 8 \\ 0.2x - 0.8 & 8 \le x \le 9 \\ 1 & X \ge 9 \end{cases}$

14. The continuous random variable X has a p.d.f given by f(x) =$\begin{cases} ax - bx^2 & 0 \le x \le 2 \\ 0 & elsewhere \end{cases}$

    If E(X) =1, find

    (i) values of a and b (a= 1.5, b= 0.75) (ii) Var(x) = 0.2

    (ii) F(X) =$\begin{cases} 0 & x < 0 \\ 0.75x^2 - 0.25x^3 & 0 \le x \le 2 \\ 1 & x \ge 2 \end{cases}$

15. The continuous random variable X has a p.d.f given by f(x) =$\begin{cases} \frac{k}{x} & 1 \le x \le 9 \\ 0 & elsewhere \end{cases}$

    Find (i) value of k =0.455, (ii) median =3 (iii) mean =3.64 (iv) Var(X) =4.95

    (v) F(X) =$\begin{cases} 0 & x < 1 \\ \frac{1}{In\,9} Inx & 1 \le x \le 9 \\ 1 & x \ge 9 \end{cases}$

16. The continuous random variable X has a p.d.f given by f(x) = $\begin{cases} \frac{20}{5^5} w^3(5 - w) & 0 \le w \le 5 \\ 0 & elsewhere \end{cases}$

    Find (i) P(2<w<5) =0.5 (ii) mean = 3.33 (iii) Var(X) = 0.794 (iv) mode = 3.5

    (v) F(X) = = $\begin{cases} 0 & w < 0 \\ \frac{w^4}{5^5}(25 - w) & 0 \le x \le 5 \\ 1 & w \ge 5 \end{cases}$

17. The continuous random variable X has a p.d.f given by f(x) = $\begin{cases} kx & 0 \le x \le 1 \\ k(4 - x^2) & 1 \le x \le 2 \\ 0 & elsewhere \end{cases}$

    Find (i) value of k = $\frac{6}{13}$ (ii) E(X) = 1.1923 (iii) Var(x) =0.1399

    (iv) F(x) = $\begin{cases} 0 & x < 0 \\ \frac{3}{13}x & 0 \le x \le 1 \\ \frac{1}{13}(24x - 2x^3 - 19) & 1 \le x \le 2 \\ 1 & x \ge 2 \end{cases}$

18. The probability density function f(x) of a random variable x takes on the form shown in the diagram below

Find

(i) Expression for f(x)

(ii) F(x), cumulative distribution function

(iii) Mean =$\frac{2}{3}$ and Var(x) =$\frac{2}{9}$

Finding f(x) from F(X)

f(x) can be obtained from; f(x) = $\frac{d}{dx}F(X)$

Example 41

The continuous random variable X has a c.d.f F(X) = $\begin{cases} 0 & x < 0 \\ \frac{x^3}{27} & 0 \le x \le 3 \\ 1 & x \ge 3 \end{cases}$

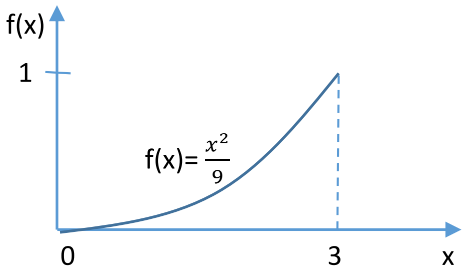
Find the probability density function f(x) and sketch f(x)

f(x) = $\frac{d}{dx}F(X)$ =$\frac{d}{dx}\left(\frac{x^3}{27}\right)$ =$\frac{3x^2}{27} = \frac{x^2}{9}$

f(x) = $\begin{cases} \frac{x^2}{9} & 0 \le w \le 3 \\ 0 & elsewhere \end{cases}$

When x = 0, f(x) = $\frac{0^2}{9} = 0$

When x = 3, f(x) = $\frac{3^2}{9} = 1$



Example 42

The continuous random variable X has a c.d.f F(X) = $\begin{cases} 0 & x < 0 \\ kx^3 & 0 \le x \le 4 \\ 1 & x \ge 4 \end{cases}$

Find

(i) Value of k

F(4) – F(0) = 1

K(4^3) = 1; k =$\frac{1}{64}$

(ii) Probability density function, f(x)

f(x) = $\frac{d}{dx}F(X)$ =$\frac{d}{dx}\frac{x^3}{64} = \frac{3x^2}{64}$

f(x) = $\begin{cases} \frac{3x^2}{64} & 0 \le w \le 4 \\ 0 & elsewhere \end{cases}$

Example 43

The continuous random variable X has a c.d.f F(X) = $\begin{cases} 0 & x < 0 \\ 2x - 2x^2 & 0 \le x \le 0.25 \\ a + x & 0.25 \le x \le 0.5 \\ b + 2x^2 - x & 0.5 \le x \le 0.75 \\ 1 & x \ge 0.75 \end{cases}$

Find

(i) Value of constants a and b

For $0 \le x \le 0.25$, F(X) =$2x - 2x^2$

F(0.25) =$2x0.25 - 2(0.25)^2$ = 0.375

For $0.25 \leq x \leq 0.5$; F(X) = a + x

F(0.25) = a + 0.25 = 0.375

a = 0.125

For $0.5 \leq x \leq 0.75$; F(X) = $b + 2x^2 - x$

F(0.75) = b + 2(0.75)^2 -0.75 = 1; b = 0.625

(ii) Probability density function f(x)

f(x) = $\frac{d}{dx}F(X)$

$$f(x) = \begin{cases} 2 - 4x & 0 \leq x \leq 0.25 \\ 1 & 0.25 \leq x \leq 0.5 \\ 4x - 1 & 0.5 \leq x \leq 0.75 \\ 0 & elsewhere \end{cases}$$

(iii) Mean = 0.375

Revision exercise 6

1. The continuous random variable X has cumulative distribution function

$$F(X) = = \begin{cases} 0 & x < 2 \\ 0.25x - 0.5 & 2 \leq x \leq 6 \\ 1 & x \geq 6 \end{cases}$$

Find the

(i) probability density function f(x); f(x)= $\begin{cases} \frac{1}{4} & 2 \leq x \leq 6 \\ 0 & elsewhere \end{cases}$

(ii) E(X) = 4 (iii) interquartile range = 2 (iv) sketch f(x)

2. The continuous random variable X has cumulative distribution function

$$F(X) = = \begin{cases} 0 & x < 0 \\ x^3 & 0 \leq x \leq 1 \\ 1 & x \geq 1 \end{cases}$$

Find (i) median (m=0.794) (ii) mean (μ =0.75)

3. The continuous random variable X has cumulative distribution function

$$F(X) = \begin{cases} 0 & x < 0 \\ x - kx^2 & 0 \leq x \leq 2 \\ 1 & x \geq 2 \end{cases}$$

Find the (i) value of k= 0.25, (ii) median (m= 0.586) (iii) variance of x (Var(x) = $\frac{2}{9}$)

(iv) probability density function; f(x) =$\begin{cases} 1 - 0.5x & 0 \leq x \leq 2 \\ 0 & elsewhere \end{cases}$

4. The continuous random variable X has cumulative distribution function

$$F(X) = \begin{cases} 0 & x < 0 \\ \frac{2x}{3} & 0 \leq x \leq 1 \\ \frac{x}{3} + k & 1 \leq x \leq 2 \\ 1 & x \geq 2 \end{cases}$$

Find (i) value of k = $\frac{1}{3}$ (ii) mean (μ = $\frac{5}{6}$) (iii) standard deviation =0.5528

(iv) P($|\mu - \sigma| < \sigma$) =0.608

(v) p.d.f; f(x) = $\begin{cases} \frac{2}{3} & 0 \leq x \leq 2 \\ \frac{1}{3} & 1 \leq x \leq 2 \\ 0 & elsewhere \end{cases}$ (vi) sketch f(x)

5. The continuous random variable X has cumulative distribution function

$$F(X) = \begin{cases} 0 & x < 1 \\ \frac{(x-1)^2}{12} & 1 \le x \le 3 \\ \frac{(14x-x^2-25)}{24} & 3 \le x \le 7 \\ 1 & x \ge 7 \end{cases}$$

Find

(i) probability density function, $f(x) = \begin{cases} \frac{1}{6}(x-1) & 1 \le x \le 3 \\ \frac{1}{12}(7-x) & 3 \le x \le 7 \\ 0 & elsewhere \end{cases}$

(ii) sketch f(x) (iii) mean of X ($\mu = \frac{11}{3}$) (iv) Var (x) = $\frac{14}{9}$ (v) median of X (m= 3.45)

(vi) P(2.8 < x < 5.2) = 0.595

6. The continuous random variable X has cumulative distribution function

$$F(X) = \begin{cases} 0 & x < -1 \\ \frac{x+1}{8} & -1 \le x \le 0 \\ \frac{3x+1}{8} & 0 \le x \le 2 \\ \frac{x+5}{8} & 2 \le x \le 3 \\ 1 & x \ge 3 \end{cases}$$

Find (i) probability density function, f(x) (ii) P($3 \le 2x \le 5$) (iii) mean and variance

7. The continuous random variable X has cumulative distribution function

$$F(X) = \begin{cases} 0 & x < 0 \\ \alpha x & 0 \le x \le 1 \\ \frac{x}{3} + \beta & 1 \le x \le 1 \\ 1 & x \ge 2 \end{cases}$$

Find (i) values of α and β ($\alpha = \frac{2}{3}$; $\beta = \frac{1}{3}$) (ii) mean ($\mu = \frac{5}{6}$) (iii) Var(X) = $\frac{19}{36}$

(iv) $P\left(X < 1.5 / X > 1\right)$= 0.4998 (v) probability density function, f(x) and sketch it

8. The continuous random variable X has cumulative distribution function

$$F(X) = \begin{cases} 0 & x < 1 \\ \frac{x^2-1}{2} - x & 1 \le x \le 2 \\ 3x - \frac{x^2}{2} & 2 \le x \le 3 \\ 1 & x \ge 3 \end{cases}$$

Find

(i) Probability density function, f(x) and sketch it

(ii) P(1.2 < x< 2.4) = 0.8

(iii) Mean (μ = 2)

9. The continuous random variable X has cumulative distribution function

$$F(X) = \begin{cases} 0 & x < 0 \\ \frac{k}{2}x^2 & 0 \le x \le 2 \\ k(6x - x^2 - 6) & 2 \le x \le 3 \\ 1 & x \ge 3 \end{cases}$$

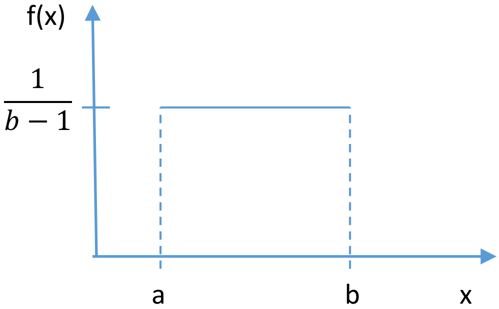
(a) Determine the value of k = $\frac{1}{3}$. Hence sketch graph of F(X)

(b) Find the probability density function.

Uniform or rectangular distribution

A continuous random variable X is said to be uniformly distributed over the interval a and b, if the p.d.f is given by f(x) = $\begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & elsewhere \end{cases}$

Graph of f(x)



Example 44

X is uniformly distributed between 6 and 9.

(i) Write the probability density function

f(x) = $\begin{cases} \frac{1}{9-6} & 6 \leq x \leq 9 \\ 0 & elsewhere \end{cases}$

(ii) Find P(7.2 < x < 8.4)

P(7.2 < x < 8.4)= $\int_{7.2}^{8.4} \frac{1}{3} dx = \frac{1}{3}[x]_{7.2}^{8.4}$ = 0.4

Example 45

X is uniformly distributed between 0 and $\frac{\pi}{2}$.

(i) Write the probability density function

f(x) = $\begin{cases} \frac{1}{\frac{\pi}{2}-0} & 0 \leq x \leq \frac{\pi}{2} \\ 0 & elsewhere \end{cases}$

(ii) Find P($\frac{\pi}{3}$ < x < $\frac{\pi}{2}$)

(iii) P($\frac{\pi}{3}$ < x < $\frac{\pi}{2}$ = $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{2}{\pi} dx = \frac{2}{\pi}[x]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{1}{3}$

Expectation of X, E(x)

E(x)= $\int_a^b xf(x)dx = \int_a^b \frac{1}{b-a} xdx = \frac{1}{2(b-a)}[x^2]_a^b = \frac{1}{2(b-a)}(b^2 - a^2) = \frac{(b+a)(b-a)}{2(b-a)} = \frac{(b+a)}{2}$

Variance of x, Var(X)

Var(x) = $\int_a^b x^2 f(x)dx - [E(x)]^2 = \int_a^b \frac{1}{b-a} x^2 dx - \left[\frac{(b+a)}{2}\right]^2 = \frac{1}{3(b-a)}[x^3]_a^b - \left[\frac{(b+a)}{2}\right]^2$

$= \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} - \left[\frac{(b+a)}{2}\right]^2 = \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} - \frac{b^2+2ab+a^2}{4}$

$= \frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12} = \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}$

Example 45

X is a rectangular distribution between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

(i) Write the probability density function; f(x) = $\begin{cases} \frac{1}{\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)} & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ 0 & elsewhere \end{cases}$

(ii) Find the mean = $\frac{(b+a)}{2} = \frac{\left(\frac{\pi}{2}+\left(-\frac{\pi}{2}\right)\right)}{2} = 0$

(iii) Find the variance of x = $\frac{(b-a)^2}{12} = \frac{\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right]^2}{12} = \frac{\pi^2}{12}$

Example 46

X is a rectangular distribution between over the interval $-3 \le x \le -1$

Find

(i) P(-2 ≤ X ≤ - 1.5) = $\int_{-2}^{-1.5} \frac{1}{2} dx = \frac{1}{2}(x)_{-2}^{-1.5}$ =$\frac{1}{4}$

(ii) Mean = $\frac{(b+a)}{2} = \frac{(-1+(-3))}{2} = -2$

(iii) Var(x) =$\frac{(b-a)^2}{12} = \frac{(-1--3)^2}{12} = \frac{1}{3}$

Revision exercise 7

1. X follows a uniform distribution with probability density function f(x) =$\begin{cases} k & 3 \le x \le 6 \\ 0 & elsewhere \end{cases}$

   Find (i) value of k = $\frac{1}{3}$ (ii) E(X) = 4.5 (iii) var(X) = 0.75 (iv) P(X>5) = $\frac{1}{3}$
2. X is distributed uniformly over -5 ≤ x ≤ -2

   Find (i) P(-4.3 ≤ X≤ -2.8) = 0.5 (ii) E(X) = -2.5 (iii) standard deviation = 0.865
3. The continuous random variable has a probability density function f(x) =$\begin{cases} \frac{1}{4} & 1 \le x \le k \\ 0 & elsewhere \end{cases}$

   Find (i) value of k = 5 (ii) P(2.1 ≤ X≤ 3.4) =0.325 (iii) E(X) = 3 (jv) Var (X) = $1\frac{1}{3}$
4. The continuous random variable has a probability density function f(x) =$\begin{cases} \frac{1}{5} & 32 \le x \le 37 \\ 0 & elsewhere \end{cases}$

   Find the probability that y lies within one standard deviation of the mean= 0.577
5. The continuous random variable X has cumulative distribution function

   $F(X) = \begin{cases} 0 & x < 2 \\ \frac{x-2}{5} & 2 \le x \le 7 \\ 1 & x \ge 7 \end{cases}$

   Find (i) E(X) = 4.5 (ii) Var(X) = $2\frac{1}{12}$
6. The continuous random variable X is uniformly distributed in the interval a ≤ x ≤ b. the lower quartile is 5 and the upper quartile is 9. Find

   (i) Values of a and b (a= 3, b = 11)

   (ii) P(6 ≤ X ≤ 7) = 0.125

   (iii) Cumulative distribution function; $F(X) = \begin{cases} 0 & x < 3 \\ \frac{x-3}{8} & 3 \le x \le 11 \\ 1 & x \ge 11 \end{cases}$
7. The number of patients visiting a certain hospital is uniformly distributed between 150 and 210

   (i) Write down the probability density function of the number of patients

$$f(x) = \begin{cases} \frac{1}{210-150} & 150 \leq x210 \\ 0 & elsewhere \end{cases}$$

(ii) Find P(170< x< 194) = 0.4